collision is rather long in comparison to $\hbar/\Delta E$, and the transition probability is small. Moreover, transitions involving a change of K are restricted by selection rules prohibiting conversion of A-ammonia into E-ammonia or viceversa. Hence energetically inelastic collisions should indeed be exceptionally rare in ammonia. This implies a drastic reduction of the contribution from the term $[\boldsymbol{J}]^{(2)}$, and correspondingly a dominance of other terms. Korving's analysis[2] indicates a strong dominance of $\boldsymbol{J}[\boldsymbol{W}]^{(2)}$. This is not very surprising, since $\boldsymbol{J}[\boldsymbol{W}]^{(2)}$ is the next simplest term in the expansion and since the dipole-dipole interaction, very far from being spherical, causes certainly many collisions without an inverse to occur, thereby allowing large contributions from the terms odd in $\boldsymbol{J}$[14]. In the authors' opinion the positive sign of the effect should not be related to the inversion of the molecule, the latter effect being much too slow to affect the collisions. Indeed, if inversion was important, the effect should be qualitatively different in heavy ammonia where inversion is 15 times slower, in contradiction to experiments.

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The Slip Problems for a Simple Gas

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Simple and accurate expressions for the velocity slip coefficient, the slip in the thermal creep, and the temperature jump coefficient are obtained by applying a variational technique to the linearized Boltzmann equation for a simple gas. Completely general forms of the boundary conditions are used, and the final results are presented in a form such that the results for any particular intermolecular force law or the gas-surface interaction law can easily be calculated. Further, it is shown that, with little extra effort, the present results can be easily extended to include the case of a polyatomic gas. It is felt that the present work, together with a recent paper in which the author has considered the solutions of the linearized Boltzmann equation for a monatomic multicomponent gas mixture, provide the desired basis for the consideration of the various slip problems associated with the polyatomic gas mixtures.

I. Introduction

Recently, in a series of papers we[1–4] have considered the slip problems for a simple monatomic gas. In these studies, we applied variational techniques (for some general remarks on the use of the variational techniques in the kinetic theory, see Refs. [4, 5]) to the linearized Boltzmann equation and we were able to obtain some simple and very accurate results. However, in this work, for the gas-wall interaction, we considered either the diffuse reflection[1, 2, 4], or the Maxwell's diffuse-specular reflection[3]. Since some recent molecular beam experiments[6, 7] have indicated that the above forms of the gas-wall interaction, are, in fact, very severe approximations of the actual forms of the gas-wall interaction, an extension of our work to include more general cases may be of considerable interest.

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Thus in this paper, for a simple gas, we derive simple and accurate expressions for the velocity slip coefficient (the Kramers' problem), the slip in the thermal creep, and the temperature slip coefficient by applying variational techniques to the linearized Boltzmann equation with completely general forms of the boundary conditions. Our final results are presented in terms of simple scalar products involving only the pertinent Chapman-Enskog solutions [8–10], the collisional invariants, and an operator A containing the effects of the gas wall interactions [11–17]. Since for any particular combination of the gas and the surface, the operator A may have some specialized form, and since the experimental and the theoretical work for the determination of these specific forms is still only in the initial stages, we have found it advisable to present our results in as general a form as possible. However, just for the sake of illustration, we do show that from our expressions, for the Maxwellian diffuse-specular reflection at the boundary, the corresponding slip terms can be deduced in a simple and almost trivial manner. For the velocity slip coefficient and the temperature slip coefficient, these results, as must be expected, are in agreement with the results for these quantities given earlier by the author. However, quite interestingly, we find that the thermal creep velocity is only weakly dependent on α (the fraction of the molecules that are diffusely reflected).

We have tried to keep our formalism and the results in a fairly general form, and an added advantage of this procedure is that our results can, in a straightforward fashion, be generalized to include even the cases of the gas mixtures and the polyatomic gases. We discuss these generalization in the Section V of this paper.

II. The Velocity Slip Coefficient (The Kramer's Problem)

We consider a semi-infinite expanse of a gas, bounded by a flat plate located at $x=0$, and lying in the $y-z$ plane. Far from the plate, the gas is maintained at a constant velocity gradient $\partial q_{\mathrm{P,\,asy}}(x)/\partial x$ normal to the plate. For small values of this gradient, we consider the linearized Boltzmann equation (to avoid unnecessary confusion, unless otherwise specified, we shall retain all quantities in their actual dimensional forms),

$$\begin{aligned} c_x \frac{\partial \Phi_{\mathrm{P}}(x,\,\mathbf{c})}{\partial x} &= L\,\Phi_{\mathrm{P}}(x,\mathbf{c}) \\ &= -\sigma(c)\;\Phi_{\mathrm{P}}(x,\mathbf{c}) \\ &\quad + \int \mathrm{d}\mathbf{c}'\, f^{(0)}(c')\; K(\mathbf{c},\mathbf{c}')\;\Phi_{\mathrm{P}}(x,\mathbf{c}') \\ &\triangleq -\sigma\,\Phi_{\mathrm{P}} + K\,\Phi_{\mathrm{P}} \end{aligned} \tag{1}$$

where L is the linearized Boltzmann collision operator, $\Phi_{\mathrm{P}}(x,\mathbf{c})$ is a measure of the perturbation in the distribution function $f_{\mathrm{P}}(x,\mathbf{c})$ from the absolute Maxwellian $f^{(0)}$, i. e. $f_{\mathrm{P}} = f^{(0)}(1+\Phi_{\mathrm{P}})$, where

$$f^{(0)} = n_0 (m/2\pi k T_0)^{3/2} \exp\{-m c^2/2 k T_0\}\,. \tag{2}$$

Here $\mathbf{c} = (c_x,\,c_y,\,c_z)$ is the velocity and $\boldsymbol{x}(x,y,z)$ is the position vector.

The boundary condition at $x=0$ is, quite generally, expressed in the form

$$\Phi_{\mathrm{P}}(0,\mathbf{c}) = A\,\Phi_{\mathrm{P}}(0,\mathbf{c}) = \int \mathrm{d}\mathbf{c}'\,\eta(-c_x')\,f^{(0)}(c')\,|c_x'|\,B(\mathbf{c},\mathbf{c}')\;\Phi_{\mathrm{P}}(0,\mathbf{c}'),\; c_x>0 \tag{3}$$

where $B(\mathbf{c},\mathbf{c}')$ is determined by the properties of the gas and the wall. It satisfies the reciprocity relation. Some details regarding the various methods for the construction of its specific form have been given by several authors [11–17]. In particular, for the Maxwell's diffuse-specular reflection, we have

$$A\,\Phi_{\mathrm{P}}(0,\mathbf{c}) = (1-\alpha)\;\Phi(0,\,-c_x,c_y,c_z) + \alpha \int \mathrm{d}\mathbf{c}'\,\eta(-c_x')\,f^{(0)}(c')\,|c_x'|\;\Phi_{\mathrm{P}}(0,\mathbf{c}')\;(1/n_0)\,(2\pi m/k T_0)^{1/2}. \tag{4}$$

In this $\eta(c_x)=1,\ c_x>0;\quad \eta(c_x)=0,\ c_x<0\,.$

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For $x \to \infty$, the function Φ_P has the form

$$\lim_{x\to\infty} \Phi_P(x, \mathbf{c}) = (m/k T_0)\, c_z\, q_{P,\mathrm{asy}}(0) + \left[\frac{m}{k T_0} c_z x + c_x c_z \Phi_p(c)\right] \partial q_{P,\mathrm{asy}}(x)/\partial x \triangleq \Phi_{P,\mathrm{asy}}(x, \mathbf{c}). \quad (5)$$

Here $c_x c_z \Phi_p(c)$ is the solution of the Chapman-Enskog viscosity equation, i. e.

$$(m/k T_0)\, c_x c_z = L(c_x c_z \Phi_p(c)). \quad (6)$$

Also, $q_P(x)$ is the mass velocity of the gas, $q_{P,\mathrm{asy}}(x)$ indicates its asymptotic value.

We find it useful to consider the Hilbert spaces in which the scalar product is defined by

$$(\varrho_1(x, \mathbf{c}), \varrho_2(x, \mathbf{c})) = \int \mathrm{d}\mathbf{c}\, \varrho_1(x, \mathbf{c})\, f^{(0)}\, \varrho_2(x, \mathbf{c}), \quad (7)$$

$$((\varrho_1(x, \mathbf{c}), \varrho_2(x, \mathbf{c}))) = \int_0^\infty \mathrm{d}x\, (\varrho_1(x, \mathbf{c}), \varrho_2(x, \mathbf{c})). \quad (8)$$

Note that

$$(m\, c_x c_z, c_x c_z \Phi_p(c)) = -\mu \quad (9)$$

where μ is the viscosity of the gas.

Now, Eq. (1), with the boundary conditions (3) and (5) is easily converted into an integral equation

$$\Phi_P(x, \mathbf{c}) = U K \Phi_P(x, \mathbf{c}) + U E A\, \Phi_P(x, \mathbf{c}) \quad (10)$$

where the operator E is defined by

$$E = \delta(x)\, \eta(c_x)\, c_x \quad (11)$$

and, the operator U is given by

$$U \varrho(x, \mathbf{c}) = \frac{1}{c_x}\left\{\eta(c_x) \int_0^x \mathrm{d}x' \exp\left[\sigma(c) \frac{x'-x}{c_x}\right] \varrho(x', \mathbf{c}) - \eta(-c_x) \int_x^\infty \mathrm{d}x' \exp\left[\sigma(c) \frac{x'-x}{c_x}\right] \varrho(x', \mathbf{c})\right\}. \quad (12)$$

It is convenient to carry out an iteration in the second term on the right-hand side of the Eq. (10), i. e., we have

$$\Phi_P(x, \mathbf{c}) = U K \Phi_P(x, \mathbf{c}) + U E A (U K + U E A)\, \Phi_P(x, \mathbf{c}).$$

Noting that $U E A (U E A)\, \Phi_P = 0$, we can write this in the form

$$\Phi_P(x, \mathbf{c}) = \mathcal{L}\, \Phi_P(x, \mathbf{c}) \quad (13)$$

where $\mathcal{L} = U K + U E A U K. \quad (14)$

Now it is convenient to introduce an operator $\mathcal{K}$ such that

$$\mathcal{K} = R K \quad (15)$$

where R is the reflection operator, i. e.

$$R f(x, \mathbf{c}) = f(x, -\mathbf{c}). \quad (16)$$

It is easily shown that the operators $\mathcal{K}$ and $(\mathcal{K}\mathcal{L})$ are, respectively, self adjoint in the spaces defined by the scalar products (7) and (8), i. e.

$$\mathcal{K}^* = \mathcal{K}, \quad (17)$$

$$(\mathcal{K}\mathcal{L})^* = \mathcal{L}^* \mathcal{K}^* = \mathcal{L}^* \mathcal{K} = \mathcal{K}\mathcal{L}. \quad (18)$$

Here we have used the facts that $K(\mathbf{c}, \mathbf{c}')$ is symmetric and rotationally invariant, and that $B(\mathbf{c}, \mathbf{c}')$ satisfies the reciprocity relation [17].

Now, we consider a function $\varrho_P(x, \mathbf{c})$ defined by

$$\Phi_P(x, \mathbf{c}) = [\varrho_P(x, \mathbf{c}) + (m/k T_0)\, c_z x + c_x c_z \Phi_p(c)]\, \partial q_{P,\mathrm{asy}}(x)/\partial x. \quad (19)$$

Substituting this in Eq. (13), we find that ϱ_P is determined by the equation

$$\varrho_P(x, \mathbf{c}) = \mathcal{L}\, \varrho_P(x, \mathbf{c}) + p_P(x, \mathbf{c}) \quad (20)$$

where

$$p_P(x, \mathbf{c}) = -\eta(c_x) \exp\left\{-\sigma(c) \frac{x}{c_x}\right\} [1 - A]\, c_x c_z \Phi_p(c). \quad (21)$$

Further, taking scalar product (7), of Eq. (1), respectively on $m c_z$ and $c_x c_z \Phi_p(c)$, and using the asymptotic solutions (5), we can show that

$$(-\mu/k T_0)\left(q_{P,\mathrm{asy}}(0) + x \frac{\partial q_{P,\mathrm{asy}}(x)}{\partial x}\right) = (c_x^2 c_z \Phi_p(c), \Phi_P(x, \mathbf{c})). \quad (22)$$

Evaluating this equation at $x=0$, and using Eqs. (19) – (21), we find that

$$q_{\mathrm{P,\,asy}}(0) = \{(c_x^2\, c_z\, \Phi_p(c), \eta(c_x)\,[1-A]\; c_x\, c_z\, \Phi_p(c)) - [(\varrho_{\mathrm{P}}(x,\mathbf{c}), p_{\mathrm{P}}^*(x,\mathbf{c}))]\}\; (k\,T_0/\mu)\; \partial q_{\mathrm{P,\,asy}}(x)/\partial x \tag{23}$$

in this

$$p_{\mathrm{P}}^*(x,\mathbf{c}) = \mathcal{K}\, p_{\mathrm{P}}(x,\mathbf{c})\,. \tag{24}$$

Hence, using the *Roussopolous variational principle* (see, e. g. Ref. [4, 5]), we find that

$$q_{\mathrm{P,\,asy}}(0) = \{(c_x^2\, c_z\, \Phi_p(c), \eta(c_x)\;[1-A]\; c_x\, c_z\, \Phi_p(c)) + F_{\mathrm{St}}[\tilde{\varrho}_{\mathrm{P}}, \tilde{\varrho}_{\mathrm{P}}^*]\}\; (k\,T_0/\mu)\; \partial q_{\mathrm{P,\,asy}}(x)/\partial x \tag{25}$$

where F_{St} is the stationary value of the functional

$$F[\tilde{\varrho}_{\mathrm{P}}, \tilde{\varrho}_{\mathrm{P}}^*] = -((\tilde{\varrho}_{\mathrm{P}}, p_{\mathrm{P}}^*)) + ((\tilde{\varrho}_{\mathrm{P}}^*, \tilde{\varrho}_{\mathrm{P}} - \mathcal{L}\tilde{\varrho}_{\mathrm{P}} - p_{\mathrm{P}})) \tag{26}$$

and ϱ_{P}^* is the solution of the adjoint equation,

$$\varrho_{\mathrm{P}}^* = \mathcal{L}^*\, \varrho_{\mathrm{P}}^* + p_{\mathrm{P}}^*\,. \tag{27}$$

Also, $\tilde{\varrho}_{\mathrm{P}}$ and $\tilde{\varrho}_{\mathrm{P}}^*$ are the trial functions, corresponding, respectively to the solutions of the Eqs. (20) and (27).

Further, a simple analysis of Eq. (27) shows that

$$\varrho_{\mathrm{P}}^*(x,\mathbf{c}) = \mathcal{K}\, \varrho_{\mathrm{P}}(x,\mathbf{c})\,. \tag{28}$$

For evaluating F_{St}, we use the asymptotic solutions of the Eqs. (20) and (27) as the trial functions, i. e. we take

$$\tilde{\varrho}_{\mathrm{P}} = a_0\, m\, c_z\,, \tag{29}$$

$$\tilde{\varrho}_{\mathrm{P}}^* = \mathcal{K}(a_0\, m\, c_z) \tag{30}$$

where a_0 is an unknown constant. Using the above in the Eqs. (25) – (26), we can write the Eq. (25) in the form

$$q_{\mathrm{P,\,asy}}(x) = \zeta\; (\partial q_{\mathrm{P,\,asy}}(x)/\partial x)]_{x=0} \tag{31}$$

where ζ is known as the velocity slip coefficient, and here it is explicitly given by

$$\zeta = \left[\,(c_x^2\, c_z\, \Phi_p(c), \eta(c_x)\,[1-A]\; c_x\, c_z\, \Phi_p(c)) + \frac{(m\, c_x\, c_z, \eta(c_x)\,[1-A]\; c_x\, c_z\, \Phi_p(c))^2}{(m\, c_x\, c_z, \eta(c_x)\,[1-A]\; m\, c_z)}\right] \frac{k\,T_0}{\mu}\,. \tag{32}$$

Generally, ζ is written in an alternative form

$$\zeta = A_{\mathrm{P}}\, l_p \tag{33}$$

where

$$l_p = \frac{2\,\mu}{\varrho_0}\left(\frac{m}{2\,k\,T_0}\right)^{1/2} \tag{34}$$

is the viscosity mean free path. ϱ_0 is the mass density.

Thus,

$$A_{\mathrm{P}} = \frac{1}{2}\, k\,T_0\, \varrho_0 \left(\frac{2\,k\,T_0}{m}\right)^{1/2} \left\{ \frac{(c_x^2\, c_z\, \Phi_p(c), \eta(c_x)\,[1-A]\; c_x\, c_z\, \Phi_p(c))}{(m\, c_x\, c_z, c_x\, c_z\, \Phi_p(c))^2} + \frac{(m\, c_x\, c_z, \eta(c_x)\,[1-A]\; c_x\, c_z\, \Phi_p(c))^2}{(m\, c_x\, c_z, c_x\, c_z\, \Phi_p(c))^2\, (m\, c_x\, c_z, \eta(c_x)\,[1-A]\; m\, c_z)} \right\}. \tag{35}$$

Note that for Maxwell's boundary conditions,

$$[1-A]\; m\, c_z = a\, m\, c_z\,, \quad [1-A]\; c_x\, c_z\, \Phi_p(c) = (2-a)\; c_x\, c_z\, \Phi_p(c)\,, \qquad (36), (37)$$

and we immediately get a result derived earlier by the present author [3]; i. e.

$$A_{\mathrm{P}} = \frac{\pi^{1/2}}{2} \left\{ (2-a)\, \frac{75}{128}\, \frac{\int_0^\infty dv\, \exp\{-v^2\}\, v^7\, \Phi_p^2(v)}{[\int_0^\infty dv\, \exp\{-v^2\}\, v^6\, \Phi_p(v)]^2} + \frac{(2-a)^2}{2\,a} \right\}; \quad \boldsymbol{v} = \left(\frac{m}{2\,k\,T_0}\right)^{1/2} \mathbf{c}\,. \tag{38}$$

Similarly, a simple calculation shows that for the model equation (4.19) of Ref. [16], p. 167, the expression (35) yields the correct result.

III. Slip in the Thermal Creep Flow

Since the influence of the accommodation on the slip in the thermal creep flow has, as yet, not been calculated accurately, the present work should be of specific interest. In this problem, we consider a semi-infinite expanse of a gas, at uniform pressure, bounded by a flat plate and lying in the $y-z$ plane. The plate, and the gas, are maintained at a constant temperature gradient $\partial T/\partial z$ in the z-direction, and for the small values of this gradient, we consider the linearized Boltzmann equation

$$c_x \frac{\partial \Phi_T(x,\mathbf{c})}{\partial x} = L\,\Phi_T(x,\mathbf{c}) - c_z\left(\frac{m}{2\,k\,T_0}\,c^2 - \frac{5}{2}\right)\frac{1}{T_0}\frac{\partial T(z)}{\partial z} \tag{39}$$

where Φ_T is a measure of the perturbation in the distribution f_T from a local Maxwellian $f^{(\mathrm{M})}$, i. e.

$$f_T(x,\mathbf{c}) = f^{(\mathrm{M})}(1+\Phi_T(x,\mathbf{c})) = f^{(0)}\left(1+\left(\frac{m}{2\,k\,T_0}\,c^2-\frac{5}{2}\right)\frac{1}{T_0}\frac{\partial T}{\partial z}\,z+\Phi_T(x,\mathbf{c})\right). \tag{40}$$

Again, the boundary equation at $x=0$, is given by an equation of the form (3), i. e.

$$\Phi_T(0,\mathbf{c}) = A\,\Phi_T(0,\mathbf{c}), \qquad c_x>0, \tag{41}$$

while for $x\to\infty$ the function $\Phi_T(x,\mathbf{c})$ has the form

$$\lim_{x\to\infty} \Phi_T(x,\mathbf{c}) = \frac{m}{k\,T_0}\,c_z\;q_{T,\,\mathrm{asy}} + \frac{1}{T_0}\frac{\partial T}{\partial z}\,c_z\,\Phi_t(c) \triangleq \Phi_{T,\,\mathrm{asy}}(x,\mathbf{c}) \tag{42}$$

where $q_{\mathrm{T,\,asy}}$, at present an unknown constant, is the asymptotic mass velocity of the gas in the z-direction, and $c_z\,\Phi_t(c)$ is the Chapman-Enskog heat conductivity solution, i. e.

$$c_z\left((m/2\,k\,T_0)\;c^2-\tfrac{5}{2}\right) = L\left(c_z\,\Phi_t(c)\right) \tag{43}$$

and it satisfies the condition

$$\left(m\,c_z,\,c_z\,\Phi_t(c)\right) = 0\,. \tag{44}$$

Also, note that

$$\left(c_z\left((m/2\,k\,T_0)\;c^2-\tfrac{5}{2}\right),\,c_z\,\Phi_t(c)\right) = -\lambda/k \tag{45}$$

where λ is the heat conductivity of the gas, and k is the Boltzmann constant.

Now, as in the previous section, we introduce a function $\varrho_T(x,\mathbf{c})$ defined by the relation,

$$\Phi_T(x,\mathbf{c}) = [\varrho_T(x,\mathbf{c}) + c_z\,\Phi_t(c)]\,\frac{1}{T_0}\frac{\partial T}{\partial z} \tag{46}$$

and note that this function is determined by the integral equation

$$\varrho_T(x,\mathbf{c}) = \mathcal{L}\,\varrho_T(x,\mathbf{c}) + p_T(x,\mathbf{c}) \tag{47}$$

where

$$p_T(x,\mathbf{c}) = -\eta(c_x)\,\exp\left(-\sigma(c)\,\frac{x}{c_x}\right)[1-A]\,c_z\,\Phi_t(c)\,. \tag{48}$$

Now we consider scalar product (7) of Eq. (39) on $m\,c_z$ and $c_x\,c_z\,\Phi_p(c)$, then using the asymptotic solution (42), we can show that

$$-\frac{\mu}{k\,T_0}\,q_{T,\,\mathrm{asy}} = \left(c_x^2\,c_z\,\Phi_p(c),\;\Phi_T(x,\mathbf{c})\right) - \left(c_x\,c_z\,\Phi_p(c),\,c_x\,c_z\,\Phi_t(c)\right)\frac{1}{T_0}\frac{\partial T}{\partial z} \tag{49}$$

Evaluating this expression at $x=0$, and using the Eqs. (46) – (47), we find that

$$q_{T,\,\mathrm{asy}} = \left\{\left(c_x^2\,c_z\,\Phi_p(c),\,\eta(c_x)\,[1-A]\,c_z\,\Phi_t(c)\right) - \left[\left(\varrho_T(x,\mathbf{c}),\,p_T^*(x,\mathbf{c})\right)\right]\right\}\frac{k}{\mu}\cdot\frac{\partial T}{\partial z}\,. \tag{50}$$

In this,

$$p_T^*(x,\mathbf{c}) = \mathcal{K}\,p_{\mathrm{P}}(x,\mathbf{c}) \tag{51}$$

where p_{P} has been defined earlier. Thus

$$q_{T,\,\mathrm{asy}} = \left\{\left(c_x^2\,c_z\,\Phi_p(c),\;\eta(c_x)\,[1-A]\,c_z\,\Phi_t(c)\right) + F_{\mathrm{St}}[\tilde{\varrho}_T,\,\tilde{\varrho}]\right\}\frac{k}{\mu}\frac{\partial T}{\partial z}\,. \tag{52}$$

Here F_{St} is the stationary value of the functional

$$F[\tilde{\varrho}_T,\,\tilde{\varrho}_T^*] = -\left((\tilde{\varrho}_T,\,p_T^*)\right) + \left((\tilde{\varrho}_T^*,\,\tilde{\varrho}_T - \mathcal{L}\,\tilde{\varrho}_T - p_T)\right) \tag{53}$$

and ϱ_T^* is the solution of the adjoint equation

$$\varrho_T^*(x,\mathbf{c}) = \mathcal{L}^*\,\varrho_T^*(x,\mathbf{c}) + p_T^*(x,\mathbf{c})\,. \tag{54}$$

Now, with the use of the properties (17) – (18) it is easily shown that

$$\varrho_T^{*}(x, \mathbf{c}) = \mathcal{K}\, \varrho_P(x, \mathbf{c}) \tag{55}$$

where ϱ_P is the solution of the Eq. (20). Again, we use the asymptotic solutions of the Eqs. (47) and (54) as the trial functions,

$$\tilde{\varrho}_T(x, \mathbf{c}) = a_0\, m\, c_z\,, \tag{56}$$

$$\tilde{\varrho}(x, \mathbf{c}) = \mathcal{K}(a_1\, m\, c_z)\,, \tag{57}$$

and after some algebra we find that

$$q_{T,\,\text{asy}} = \Big[\big(c_x c_z \Phi_p(c), \eta(c_x)\, c_x [1-A]\, c_z \Phi_t(c)\big) + \frac{\big(m c_x c_z, \eta(c_x)[1-A]\, c_x c_z \Phi_p(c)\big)\big(m c_x c_z, \eta(c_x)[1-A]\, c_x \Phi_t(c)\big)}{\big(m c_x c_z, \eta(c_x)[1-A]\, m c_z\big)} \Big] \frac{k T_0}{\mu} \frac{1}{T_0} \frac{\partial T}{\partial z} \tag{58}$$

which then is the slip velocity in the thermal creep flow. Usually, this expression is written in an alternative form [4],

$$q_{T,\,\text{asy}} = \frac{1}{2} A_T \left(\frac{2 k T_0}{m}\right)^{1/2} \frac{1}{T_0} \frac{\mathrm{d}T}{\mathrm{d}z}\, l_t \tag{59}$$

where

$$l_t = \frac{4}{5} \frac{\lambda}{p} \left(\frac{m T_0}{2k}\right)^{1/2} = -\frac{4}{5} \frac{1}{n_0} \left(\frac{m}{2 k T_0}\right)^{1/2} \left(c_z \left(\frac{m}{2 k T_0} c^2 - \frac{5}{2}\right), c_z \Phi_t(c)\right) \tag{60}$$

is a thermal mean free path [2–4]. A_T is a dimensionless number and is given by (note, $p = n_0 k T_0$);

$$A_T = -\frac{5}{2} p \frac{k}{\lambda} \left\{ \frac{\big(c_x c_z \Phi_p(c), \eta(c_x)\, c_x [1-A]\, c_z \Phi_t(c)\big)}{\big(c_x c_z \Phi_p(c), m c_x c_z\big)} + \frac{\big(m c_x c_z, \eta(c_x)[1-A]\, c_x c_z \Phi_p(c)\big)\big(m c_x c_z, \eta(c_x)[1-A]\, c_x \Phi_t(c)\big)}{\big(m c_x c_z, c_x c_z \Phi_p(c)\big)\big(m c_x c_z, \eta(c_x)[1-A]\, m c_z\big)} \right\}. \tag{61}$$

Now let us consider the specific case of the Maxwellian diffuse-specular reflection at the boundary. Then

$$[1-A]\, m c_z = \alpha\, m c_z\,, \quad [1-A]\, c_z \Phi_t(c) = \alpha\, c_z \Phi_t(c)\,, \qquad (62), (63)$$

$$[1-A]\, c_x c_z \Phi_p(c) = (2-\alpha)\, c_x c_z \Phi_p(c) \tag{64}$$

and we have $(\alpha \neq 0)$,

$$A_T = -\frac{5}{2} p \frac{k}{\lambda} \left\{ \frac{\big(c_x c_z \Phi_p(c), \eta(c_x)\, c_x c_z \Phi_t(c)\big)}{\big(c_x c_z \Phi_p(c), m c_x c_z\big)} \alpha + \frac{\big(m c_x c_z, \eta(c_x)\, c_z \Phi_t(c)\big)}{\big(m c_x c_z, \eta(c_x)\, m c_z\big)} \frac{2-\alpha}{2} \right\}. \tag{65}$$

Further, specialising it to the case of the Maxwell molecules

$$c_z \Phi_t(c) = -\left(\frac{m}{2 k T_0}\right)^{1/2} l_t\, c_z \left(\frac{m}{2 k T_0} c^2 - \frac{5}{2}\right), \tag{66}$$

$$c_x c_z \Phi_p(c) = -\frac{m}{k T_0} \frac{\mu}{p} c_x c_z \tag{67}$$

and we find that

$$A_T(\alpha) = \frac{1}{2} + \frac{\alpha}{4} = A_T(1) - \frac{1}{4}(1-\alpha). \tag{68}$$

Thus, our calculations indicate that A_T has only a weak dependence on α. Quite interestingly, for the model boundary condition (4.19) of Ref. [16], p. 167, we find that for all intermolecular force laws

$$A_T(\alpha) = A_T(1)\,, \tag{69}$$

i. e. for this model boundary condition the slip velocity does not depend on the accommodation coefficient. Of course, for this model boundary condition, the above result could be directly shown by simply noting that there is no shear stress in the gas.

IV. The Temperature Jump Coefficient

Here we consider a semi-infinite expanse of a gas, bounded by a flat plate located at $x=0$, and lying in the $y-z$ plane. Far from the plate, the gas is maintained at a constant temperature gradient $\partial T_{\text{asy}}(x)/\partial x$, normal to the plate. For small values of this gradient, we consider the linearized Boltzmann equation (we shall again use the subscript T here; no confusion should arise as the solution of this problem and the solution of the creep problem are quite unrelated),

$$c_x \frac{\partial \Phi_T(x, \mathbf{c})}{\partial x} = L\, \Phi_T(x, \mathbf{c})\,, \tag{70}$$

where $\Phi_T(x, \mathbf{c})$ is a measure of the perturbation in the distribution function $f_T(x, \mathbf{c})$ from an absolute Maxwellian $f^{(0)}(c)$ corresponding to the temperature of the wall, i. e.

$$f_T(x, \mathbf{c}) = f^{(0)}(c)\,(1 + \Phi_T(x, \mathbf{c}))\,. \tag{71}$$

Again, the boundary condition at $x=0$ is given by

$$\Phi_T(0, \mathbf{c}) = A\,\Phi_T(0, \mathbf{c}), \qquad c_x > 0\,, \tag{72}$$

While for $x \to \infty$, the function Φ_T has the form

$$\lim_{x\to\infty} \Phi_T(x, \mathbf{c}) = \left(\frac{m}{2\,k\,T_0}\,c^2 - \frac{5}{2}\right)\frac{T_{\text{asy}}(0) - T_0}{T_0} + \left(\frac{m}{2\,k\,T_0}\,c^2 - \frac{5}{2}\right)\frac{1}{T_0}\,\frac{\partial T_{\text{asy}}(x)}{\partial x} + c_x\,\Phi_t(c)\,\frac{1}{T_0}\,\frac{\partial T_{\text{asy}}(x)}{\partial x} \triangleq \Phi_{T,\,\text{asy}}(x, \mathbf{c})\,. \tag{73}$$

Here $T(x)$ is the temperature of the gas, $T_{\text{asy}}(x)$ indicates the value of its asymptotic component. $c_x\,\Phi_t(c)$ is the Chapman-Enskog heat conductivity solution, and has been disussed in the previous section.

Now we introduce a function $\varrho_T(x, \mathbf{c})$ defined by the relation

$$\Phi_T(x, \mathbf{c}) = \left[\varrho_T(x, \mathbf{c}) + \left(\frac{m}{2\,k\,T_0}\,c^2 - \frac{5}{2}\right)x + c_x\,\Phi_t(c)\right]\frac{1}{T_0}\,\frac{\partial T_{\text{asy}}(x)}{\partial x} \tag{74}$$

and note that this is determined by the integral equation

$$\varrho_T(x, \mathbf{c}) = \mathcal{L}\,\varrho_T(x, \mathbf{c}) + p_T(x, \mathbf{c})\,, \tag{75}$$

in which

$$p_T(x, \mathbf{c}) = -\eta(c_x)\,\exp\left\{-\sigma(c)\,\frac{x}{c_x}\right\}\,[1 - A]\,c_x\,\Phi_t(c)\,. \tag{76}$$

Again, we consider scalar product (7) of Eq. (70) on $((m/2\,k\,T_0)\,c^2 - \frac{5}{2})$ and $c_x\,\Phi_t(c)$. Then, using the asymptotic solution (73), we can show that

$$-\frac{\lambda}{k}\left(\frac{T_{\text{asy}}(0) - T_0}{T_0} + x\,\frac{1}{T_0}\,\frac{\partial T_{\text{asy}}(x)}{\partial x}\right) = (c_x\,\Phi_t(c),\, c_x\,\Phi_T(x, \mathbf{c}))\,. \tag{77}$$

Evaluating this expression at $x=0$, and using Eqs. (74) and (75), we find that

$$\frac{T_{\text{asy}}(0) - T_0}{T_0} = \{(c_x{}^2\,\Phi_t(c),\, \eta(c_x)\,[1-A]\,c_x\,\Phi_t(c)) + ((\varrho_T, p_T^*))\}\,\frac{k}{\lambda}\,\frac{1}{T_0}\,\frac{\partial T_{\text{asy}}(x)}{\partial x} \tag{78}$$

where

$$p_T^* = \mathcal{K}\,p_T\,. \tag{79}$$

Thus

$$\frac{T_{\text{asy}}(0) - T_0}{T_0} = \{(c_x{}^2\,\Phi_t(c),\, \eta(c_x)\,[1-A]\,c_x\,\Phi_t(c)) - F_{\text{St}}[\tilde{\varrho}_T, \tilde{\varrho}_T^*]\}\,\frac{k}{\lambda}\,\frac{1}{T_0}\,\frac{\partial T_{\text{asy}}(x)}{\partial x}\,. \tag{80}$$

In this F_{St} is the stationary value of the functional

$$F[\tilde{\varrho}_T, \tilde{\varrho}_T^*] = -((\varrho_T, p_T^*)) + ((\tilde{\varrho}_T^*, \tilde{\varrho}_T - \mathcal{L}\,\tilde{\varrho}_T - p_T)) \tag{81}$$

and ϱ_T^* is the solution of te adjoint equation

$$\varrho_T^*(x, \mathbf{c}) = \mathcal{L}^*\,\varrho_T^*(x, \mathbf{c}) + p_T^*(x, \mathbf{c})\,. \tag{82}$$

With the use of the properties (17) and (18), again it is easily shown that ϱ_T^* is simply related to the function ϱ_T, i. e. we find

$$\varrho_T^*(x, \mathbf{c}) = \mathcal{K}\,\varrho_T(x, \mathbf{c})\,. \tag{83}$$

For evaluating F_{St}, we choose asymptotic solutions of the Eqs. (75) and (82) as the trial functions,

$$\tilde{\varrho}_T(x, \mathbf{c}) = a_0[(m/2\,k\,T_0)\,c^2 - \tfrac{5}{2}]\,, \tag{84}$$

$$\tilde{\varrho}_T^*(x, \mathbf{c}) = \mathcal{K}\,(a_0[(m/2\,k\,T_0)\,c^2 - \tfrac{5}{2}]) \tag{85}$$

where a_0 is an unknown constant. Thus we find that

$$T_{\text{asy}}(x) = T_0 + \mathcal{G}\left[\frac{\partial T_{\text{asy}}(x)}{\partial x}\right]_{x=0} \tag{86}$$

where $\mathcal{G}$, known as the temperature Jump coefficient, is given by

$$\mathcal{G} = \left[\left(c_x \Phi_t(c), \eta(c_x)\, c_x[1-A]\, c_x \Phi_t(c)\right) + \frac{\left(c_x\left(\frac{m}{2\,k\,T_0}\,c^2 - \frac{5}{2}\right), \eta(c_x)\,[1-A]\; c_x \Phi_t(c)\right)^2}{\left(c_x\left(\frac{m}{2\,k\,T_0}\,c^2-\frac{5}{2}\right), \eta(c_x)\,[1-A]\left(\frac{m}{2\,k\,T_0}\,c^2-\frac{5}{2}\right)\right)}\right] \frac{k}{\lambda}\,. \quad (87)$$

Usually, $\mathcal{G}$ is written in an alternative form $\mathcal{G} = \varepsilon'\, l_t\,,$ (88)

where, l_t has been defined earlier, ε' is a dimensionless number [2, 3] and is given by

$$\varepsilon' = \frac{5}{4}\, n_0 (2\,k\,T_0/m)^{1/2} \left\{ \frac{(c_x^2\, \Phi_t(c), \eta(c_x)\,[1-A]\; c_x\, \Phi_t(c))}{(c_x[(m/2\,k\,T_0)\, c^2 - \frac{5}{2}], c_x\, \Phi_t(c))^2} + \frac{(c_x[(m/2\,k\,T_0)\, c^2-\frac{5}{2}], \eta(cx)\,[1-A]\; c_x\,\Phi_t(c))^2}{(c_x[(m/2\,k\,T_0)\,c^2-\frac{5}{2}], c_x\,\Phi_t(c))\,(c_x[(m/2\,k\,T_0)\,c^2-\frac{5}{2}]), \eta(cx)\,[1-A]((m/2\,k\,T_0)\,c^2-\frac{5}{2}))} \right\}. \quad (89)$$

Note that for the Maxwellian boundary conditions,

$$[1-A]\; c_x\, \Phi_t(c) = (2-\alpha)\; c_x\, \Phi_t(c)\,, \quad (90)$$

$$[1-A]\left(\frac{m}{2\,k\,T_0}\,c^2 - \frac{5}{2}\right) = \alpha\left(\frac{m}{2\,k\,T_0}\,c^2 - 2\right) \quad (91)$$

and the expression (89) immediately reduces to

$$\varepsilon' = \frac{5}{8}\,\pi^{1/2} \left\{ (2-\alpha)\,\frac{9}{16}\, \frac{\int_0^\infty \mathrm{d}v\, \exp\{-v^2\}\, v^5\, \Phi_t^2(v)}{[\int_0^\infty \mathrm{d}v\, \exp\{-v^2\}\, v^6\, \Phi_t(v)]^2} + \frac{(2-\alpha)^2}{2\,\alpha} \right\}, \quad (92)$$

which is precisely the result obtained earlier by the present author [see Eq. (3.14), Ref. [3]]. In a similar way, the results for any form of the boundary conditions are easily deduced.

V. The Case of a Polyatomic Gas

It is well known that for a polyatomic gas, in place of the Boltzmann equation, one should use the Wang-Chang-Uhlenbeck [18] (WCU) equation (or for more complicated cases the Waldmann-Snider [19] equation). The distribution function now depends also on ε_i, the internal energy corresponding to the i-th state, i. e. $f = f(x, \mathbf{c}, \varepsilon_i)$. The linearized WCU equation can be written in the form

$$\begin{aligned} c_x \frac{\partial \Phi(x,\mathbf{c},\varepsilon_i)}{\partial x} &= -\sigma(c,\varepsilon_i)\; \Phi(x,\mathbf{c},\varepsilon_i) \\ &\quad + \sum_j \int \mathrm{d}\mathbf{c}'\, f_0(c',\varepsilon_j)\, K(\mathbf{c},\varepsilon_i;\mathbf{c}',\varepsilon_j)\; \Phi(x,\mathbf{c}',\varepsilon_j) \\ &= L^{\mathrm{WCU}}\, \Phi(x,\mathbf{c},\varepsilon_i) \qquad (93) \\ &\triangleq -\sigma^{\mathrm{WCU}}\, \Phi(x,\mathbf{c},\varepsilon_i) + K^{\mathrm{WCU}}\, \Phi(x,\mathbf{c},\varepsilon_i)\,, \end{aligned}$$

where $\Phi(x, \mathbf{c}, \varepsilon_i)$ is a measure of the perturbation in the distribution function f from the absolute Maxwellian $f^{(0)}(c, \varepsilon_i)$, i. e. $f = f^{(0)}(1+\Phi)$ [no doubt if the linearization is about a local Maxwellian, then as in the Eq. (39), inhomogeneous terms may appear in the Eq. (93); similar considerations hold for the boundary conditions]. Note that $K(\mathbf{c}, \varepsilon_i, \mathbf{c}', \varepsilon_j)$ is symmetric and rotationally invariant.

Again, corresponding to the Eq. (3), for the boundary conditions we have [17]

$$\begin{aligned} \Phi(0,\mathbf{c},\varepsilon_i) &= A\; \Phi_{\mathrm{P}}(0,\mathbf{c},\varepsilon_i) \\ &= \sum_j \int \mathrm{d}\mathbf{c}'\, \eta(-c_x')\, f^{(0)}(c',\varepsilon_j) \qquad (94) \\ &\quad \cdot |c_x'|\, B(\mathbf{c},\varepsilon_i;\mathbf{c}',\varepsilon_j)\; \Phi(0,\mathbf{c}',\varepsilon_j)\,. \end{aligned}$$

While for $x \to \infty$ we use the Chapman-Enskog expansions appropriate to a polyatomic gas [18]. Thus, for the collisional invariants now we have 1, $m\,\mathbf{c}$, and $[(m/2\,k\,T_0)\,c^2 - \frac{5}{2} + \varepsilon_i - \bar{\varepsilon}]$. Also, the Chapman-Enskog viscosity and the heat conductivity solutions are, respectively, determined from the equations [18]

$$(m/k\,T_0)\; c_x\, c_z = L^{\mathrm{WCU}}\left(c_x\, c_z\, \Phi_p(c,\varepsilon_i)\right) \quad (95)$$

and

$$c_x[(m/2\,k\,T_0)\; c^2 - \tfrac{5}{2} + \varepsilon_i - \bar{\varepsilon}] = L^{\mathrm{WCU}}\left(c_x\, \Phi_t(c,\varepsilon_i)\right). \quad (96)$$

Now it is easily shown that the procedures employed for the calculation of the slip terms for a simple gas, can be equally well adopted for the calculation of the slip terms for a polyatomic gas. Only now, instead of the Eqs. (7) and (8), we must use the Hilbert spaces that are defined by the

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scalar products

$$(\varrho_1(x, \mathbf{c}, \varepsilon_i), \varrho_2(x, \mathbf{c}, \varepsilon_i)) = \sum_i \int \mathrm{d}\mathbf{c}\, \varrho_1(x, \mathbf{c}, \varepsilon_i)\, f^{(0)}(c, \varepsilon_i)\, \varrho_2(x, \mathbf{c}, \varepsilon_i), \tag{97}$$

$$\left((\varrho_1(x, \mathbf{c}, \varepsilon_i), \varrho_2(x, \mathbf{c}, \varepsilon_i))\right) = \int_0^\infty \mathrm{d}x\, (\varrho_1(x, \mathbf{c}, \varepsilon_i), \varrho_2(x, \mathbf{c}, \varepsilon_i)). \tag{98}$$

Thus, it is easily shown that in the expressions (32), (58), (87) (and in the other expressions derived from these), if we replace $[(m/2\,k\,T_0)\,c^2-\frac{5}{2}]$ by $[(m/2\,k\,T_0)\,c^2-\frac{5}{2}+\varepsilon_i-\bar{\varepsilon}]$; $\Phi_p(c)$ by $\Phi_p(c, \varepsilon_i)$, and $\Phi_t(c)$ by $\Phi_t(c, \varepsilon_i)$, and interpret the scalar products in the sense of the Eq. (97), then these expressions give the slip terms for the polyatomic gases too. It is clear that some similar extensions should be valid for the gas mixtures also.

As an interesting consequence of this generalization, it is a simple matter to show that, at least for the first order solutions [18] of the Eqs. (95) and (96), and the diffuse reflection, the slip in the thermal creep is dependent only on the translational part of the thermal conductivity. This result is in qualitative agreement with the results derived via the use of some other theories [20, 21]. However, quantitatively, we feel that the results reported here should be more accurate.

We have considered some simple calculations for the temperature jump coefficient also, and for some special model baundary conditions our results immediately reduce to a result (derived via the use of the integrodifferential form of the variational principle) to be reported shortly [22]. The details of such calculations will not be given here.

VI. Discussions and Conclusions

We have shown that our previous work on the slip problems can be easily extended to include the general forms of the boundary conditions. Also, we have presented the final results in such a form that for a simple gas, the numerical values for a particular intermolecular force law and the gas-wall interaction law can be easily calculated.

Our formalism has an added advantage that our results, in a simple manner, can be generalised to include the case of a polyatomic gas. We believe that in view of the present work, and a recent paper in which we considered the soultion of the problems associated with multicomponent monatomic gas mixtures [23], it should be an easy matter to treat the problems associated with the polyatomic gas mixtures. Also, the strikingly simple and similar forms of our results suggest that, perhaps, it should be possible to construct a simple general rule for the calculation of the slip terms. These aspects of the problem will be investigated in a later paper.

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